# VIBRATION TEST FORCE LIMITS DERIVED FROM FREQUENCY SHIFT METHOD

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## Abstract

The two-degree-of-freedom system (TDFS) method used to derive force specifications for force limited vibration tests at JPL is reviewed and some limitations of the method and of the simple TDFS model are discussed. A new improved "frequency shift" force prediction method, is developed and applied to a more complex model where the load and source have both residual and modal masses.

## Background

During the past three years, force limiting has been utilized in ten JPL vibration tests to prevent overtesting of flight hardware.[1,2] In force limited vibration tests, the shaker force is limited to the predicted maximum flight forces. In recent JPL tests, the maximum flight forces have been predicted using the two-degree-of-freedom system (TDFS) shown in the upper right corner of Fig. 1. The spectrum of the maximum force ($S_{ff}$) was calculated from Eq. 1:3

$$S_{ff} = M_2^2 \, S_{aa} \, E(x_2^2)/E(x_1^2) \qquad (1)$$

where: $M_2$ is the load oscillator mass, $S_{aa}$ is the spectrum of the source acceleration, and $E(x_2^2)/E(x_1^2)$ is the ratio of load to source mean-square responses from TDFS random vibration analysis.[4] The force spectrum in Eq.1, normalized by the load oscillator mass squared and the source acceleration spectrum, is plotted, as open symbols, against the ratio of the load mass to the source mass in Fig. 1. For small values of the ratio of load to source mass, the load has little effect on the source and the normalized force spectrum asymptote is $Q^2/2$, where Q is the amplification factor, i.e. the reciprocal of twice the critical damping ratio. For large values of mass ratio, the normalized force spectrum approaches unity, i.e. there is no amplification regardless of the Q value.

To apply Eq. 1 to a vibration test, it is necessary to identify the source oscillator acceleration as the acceleration test specification; and to determine the masses of the source and load oscillators from the properties of the actual distributed systems. The two oscillators in Fig. 1 represent coupled resonant modes of the source and load in each frequency band, e.g. one-third octaves, so the oscillator masses depend on frequency. These masses represent the mass-like, as opposed to damping or stiffness, properties of the modes as seen at the source/load connection or drive point. In the JPL force limited vibration testing applications, the oscillator mass has been taken as the residual mass (the sum of the effective masses of all modes with resonances above the excitation frequency) from FEM analyses 5, or alternately as the smoothed frequency response function (FRF) of the ratio of drive point force to acceleration measured with a shaker or impact hammer.

The single mass models of the source and load in Fig. 1 and the related Eq. 1 have inherent conceptual difficulties and are limited in their capability to accurately represent the force contributions of both resonant and nonresonant modes. These deficiencies and the desire to predict peak rather than mean-square forces provided the motivation to develop an improved force prediction method.

## Frequency Shift Method of Predicting Maximum Force

This report discusses an improved method for calculating the maximum interface force between two vibratory systems for the purpose of defining force limits for vibration tests. Some of the rationale for the method was provided by Smallwood.[6] For both the coupled source and load flight configuration and the isolated load vibration test configuration, the interface force autospectrum $S_{ff}$ is related to the interface acceleration autospectrum $S_{aa}$ by Eq. 2, which is F=MA for random vibration:

$$S_{ff}(w) = |\mathbf{M}_2(w)|^2 S_{aa}(w) \qquad (2)$$

where: $M_2$ is the load dynamic mass, i.e. the FRF's (magnitude and phase) of the ratio of the drive point force to acceleration, which is the same for both configurations. (Bold type is used herein to indicate a FRF.) The term "dynamic" mass is used here to include the complete dynamic response including resonance and stiffness effects not included by the previously used terms "oscillator" mass and "effective" mass. The radian frequency w dependence is shown explicitly in Eq. 2 to emphasize that the relation between force and acceleration applies at each frequency.

The application of Eq. 2 to a coupled source and load system is illustrated in the Fig. 2 FRF curves, which are for the simple TDFS model shown in Fig. 1 with identical oscillators and unit masses and excitation.[7] Fig. 2a shows the magnitude of the load dynamic mass, which peaks at the load natural frequency $f_o$ with an amplitude of Q, times the input. Fig. 2 b and c show the magnitude of the coupled system interface acceleration and force, respectively. Eq. 2 may be used to calculate frequency point by point the force in Fig. 2c from the load dynamic mass in Fig. 2a and the acceleration in Fig. 2b. For example, applying Eq. 2 at the 0.62 Hz coupled system resonance frequency in Fig. 2, the load dynamic mass of approximately 1.6 times the peak acceleration of 50 equals the peak force of 80, and at 1.62 Hz the load dynamic mass of approximately 0.6 times the peak acceleration of 8 equals the peak force of 5. Notice from Fig. 2 that both the interface force and acceleration peak at the coupled system natural frequencies, $0.62f_o$ and $1.62f_o$ for the identical Fig.1 oscillators. It can be shown that this is a general result, by expresing the interface force and acceleration in terms of the numerators and denominators of the drive point and transfer dynamic masses.

As a first example of the improved method of calculating force limits, the evaluation of the maximum force for the simple TDFS shown in Fig. 1 is revisited. The characteristic equation for a dynamic absorber, from Den Hartog[8], is used to calculate the coupled system resonance frequencies ($w_+$ and $w_-$) for the oscillators in Fig. 1:

$$(w/w_o)^2 = (1+u/2) \pm (u+u^2/4)^{0.5} \qquad (3)$$

where: $w_o$ is the resonance frequency of the load oscillator and u is the ratio of load to source masses ($M_2/M_1$). The peak in the normalized interface force spectrum at each of the two resonance frequencies is calculated from the magnitude squared of the load dynamic mass using Eq. 2 which for the TDFS in Fig.1 becomes:

$$S_{ff}/(S_{aa} M_2^2) = (1+\beta^2/Q^2)/[(1-\beta^2)^2+\beta^2/Q^2] \qquad (4)$$

where $\beta$ is the ratio of excitation frequency w to load resonance frequency $w_o$.

The new method is called the "frequency shift" method because the maximum forces in the coupled system are calculated by evaluating the load dynamic mass of Eq. 4 at the coupled system, or shifted, resonance frequencies $w_+$ and $w_-$ from Eq. 3 instead of at its peak at the uncoupled load resonance frequency $w_o$. The ratio of the coupled system maximum force to the force in a conventional vibration test, the so called "knock down" factor, is equal to the ratio of the value of the load dynamic mass at the shifted frequencies to the peak value at the uncoupled load resonance frequency.

Figure 1 compares the spectral peak value of the normalized force spectrum (the greater of the values at $w_+$ and $w_-$) calculated from Eq. 4 with the maximum normalized force spectrum calculated from Eq. 1 using the mean-square response ratio. Notice that for large values of the ratio of load to source mass the two calculations are in agreement For small values of the mass ratio, the peak result is a factor of two higher.

## Calculation of Maximum Force for Residual and Modal Mass Model

Herein, the frequency shift method is used to calculate the maximum force for a more complex TDFS model in which the source and load each have two masses to represent both the residual and modal mass of a continuous system. It is assumed that the acceleration specification correctly envelopes the higher of the two acceleration peaks of the coupled TDFS system. The calculation of the normalized maximum force requires accounting for the ratio of the acceleration peaks at the two coupled system resonance frequencies. Calculation of the maximum force for this new model also necessitates a tuning analysis, conducted in 3% increments, considering different ratios of the load and source uncoupled resonance frequencies. In addition, the complexity of the model requires that the results be presented in parametric curves for different ratios of modal to residual mass for both the source and load.

Figure 3a shows a model of a source and load in which each mode may be represented as a single-degree-of-freedom system attached to the connection interface. (This type of model is sometimes called an asparagus patch model.) Derivation of this type of model from a FEM analysis requires normalizing the modes so that the inertial forces equal the reaction forces at the interfaces

When this model is excited at the interface at a frequency near the resonance frequency $w_n$ of the $n^{th}$ mode, the model may be simplified to that in Fig. 3b, where $m_n$ is the modal mass of the $n^{th}$ mode and $M_n$ is the residual mass, i.e. the sum of the masses of the $n^{th}$ and all higher resonance frequency modes. Finally, Fig. 3c shows the coupled system model which results from coupling a residual and modal mass model of both the source and load. The ratio of modal to residual mass is $a_1 = m_1/M_1$ for the source and $a_2 = m_2/M_2$ for the load; the ratio of load to source resonance frequency is $\Omega = w_2/w_1$; and the ratio of load to source residual mass is $\mu = M_2/M_1$. The maximum force for a model similar to that in Fig. 3c was calculated by Smallwood[6] for the special case of equal modal and residual masses for both the source and load ($a_1 = a_2 = 1$) and a ratio of load to source resonance frequency of square root of two ($\Omega = 2^{0.5}$).

The undamped resonance frequencies of the coupled system in Fig. 3c are solutions of:

$$(1-\beta_1^2)(1-\beta_2^2) + a_1(1-\beta_2^2) + \mu(1-\beta_1^2)(1-\beta_2^2) + \mu a_2(1-\beta_1^2) = 0 \quad (5)$$

where $\beta_1 = w/w_1$, $\beta_2 = w/w_2$ and $W_1$ & $W_2$ are the uncoupled system resonance frequencies.

The $w_+$ and $w_-$ resonance frequencies are found from the quadratic equation solution:

$$w_+ \;\&\; w_- = -B/2 \pm (B^2-4C)^{0.5}/2 \quad (6)$$

where:

$$B = [(1+\mu+a_1)/\Omega^2 + (1+\mu+\mu a_2)]/( \;\; +\mu)$$

and $C = (1+\mu+a_1+\mu a_2)/(1+\mu)\Omega^2$

The magnitude squared of the load dynamic mass for the residual and modal mass model is:

$$|M_2|^2/M_2^2 = \{[(1\beta_2^2)+a_2]^2+\beta_2^2(1+a_2)^2/Q_2^2\}/[(1-\beta_2^2)^2+\beta_2^2/Q_2^2] \quad (7)$$

where $Q_2$ is the amplification factor of the load resonances. Substituting the frequencies from Eq. 6 into Eq. 7, yields the ratio of the interface force spectral density peak to interface acceleration spectral density peak at each of the two coupled system resonance frequencies.

The desired result is the ratio of the larger of the two force spectral density peaks to the larger of the two acceleration spectral density peaks, the former being the desired force limit and the latter corresponding to the acceleration specification. A problem is that the peak acceleration and peak force do not necessarily occur at the same frequency, e.g. the peak acceleration may be at the higher of the two coupled system resonance frequencies while the peak force occurs at the lower of the two frequencies. This problem is particularly pronounced when the resonance frequencies of the load and source are approximately equal, $\Omega$ near unity.

In order to obtain the desired result, it is necessary to calculate the ratio of the two acceleration spectral density peaks of the coupled system response and this ratio depends on how the system is excited. For example, one value of the acceleration ratio is obtained if it is assumed that the free acceleration of the residual mass of the source system is constant with frequency, and a different value is obtained if it is assumed that the modal mass of the source system is excited with a force which is constant with frequency. Herein the latter is assumed, since it is thought to be more typical. Once the acceleration spectrum peak ratio is obtained, the dynamic masses in Eq. 7 are scaled by multiplying the dynamic mass at the frequency corresponding to the lower acceleration peak by the ratio of the lower to the higher acceleration and by multiplying the dynamic mass at the other frequency by unity. Finally, the larger of the two thus scaled dynamic masses is used as the ratio of maximum force to acceleration specification.

The magnitude squared of the ratio of coupled system interface acceleration A to the free acceleration $A_{lo}$ of the residual mass of the source is:

$$|A/A_{1o}|^2 = |M_1/(M_1+M_2)|^2 \quad (8)$$

where Ml is obtained from Eq. 7 by replacing the subscript 2 by 1. The values of the above ratio at each of the two coupled system resonance frequencies is obtained by substituting the coupled system resonance frequencies from Eq. 6 into Eq. 8. However, as previously discussed, the free acceleration $A_{lo}$ will not generally be the same at the two resonance frequencies. For an external force $F_e$ acting on the source modal mass $m_1$ the magnitude squared of the free acceleration $A_{1o}$ is:

$$|A_{1o}/(F_e/m_1)|^2 = \beta_1^4(1+\beta_1^2/Q_1^2)/\{[(1-\beta_1^2)(1-\beta_1^2/a_1)-1]^2+\beta_1^6(1+1/a_1)^2/Q_1^2\} \quad (9)$$

Combining Eq. 8 and Eq. 9 yields the desired ratio of interface acceleration to external force. Assuming that the external force is the same at the lower and upper coupled system resonance frequencies, the interface acceleration ratio at the two frequencies is calculated by evaluating Eqs. 8 and 9.

The final step in the derivation of the maximum force is to vary the ratio, $\Omega = w_2/w_l$, of the resonance frequencies of the load to the source to insure that the maximum value of the interface force is found for all the mass and damping combinations considered. A tuning analysis was conducted in which the value of the frequency ratio squared $\Omega^2$ was varied by 1/16ths from 1 to 37/16ths, which corresponds to 3% increments in frequency ratio. The maximum values of the force spectra, normalized by the maximum values of the acceleration spectra and the load residual mass squared, are listed in Table 1 for a load amplification values $Q_2$ of 20. (Results for other Q's are available from author.) The maximum forces are rounded to whole numbers. The tuning frequency ratio squared in 16ths, which resulted in the maximum forces, are identified by the digits to the right of the decimal in Table 1.

## Use of Table to Predict Force Limits

To use the new force limit results, it is necessary to have both the residual and modal masses of the source and load as a function of frequency, either from an FEM analysis, or from a test, or from both. FEM analyses with the modal masses normalized as in Wada[5] provide both the modal and residual effective masses. Where tests have been used to measure the effective mass, with either a shaker or tap hammer instrumented to measure force, the smoothed FRF has been taken as the residual mass in deriving the force specifications for past JPL test projects.[7,8] The modal mass is by definition the negative change in the residual mass at the resonance frequencies, but some work is needed to develop a procedure for deriving the modal masses from test data. Currently, it is recommended that force limits be calculated using both the simple TDFS model and the new residual and modal mass model, and that the higher force calculated in each frequency band be used to limit the force in the test.

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| AMPLIFICATION ACTOR | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| RESIDUAL MASS RATIO (M2/M1) | | 0.001 | 0.003 | 0.01 | 0.03 | 0.1 | 0.3 | 1 | 3 | 10 |
| RATIO OF MODAL TO RESIDUAL MASS | | | | | | | | | | |
| /M | Load 2/M | | | | | | | | | |
| 1 | 0.7 | 187.32 | 157.32 | 102.32 | 64.31 | 34.3 | 21.28 | 17.24 | 15.22 | 14.22 |
| 1 | 0.5 | 100.32 | 89.32 | 63.32 | 42.31 | 24.3 | 15.28 | 12.24 | 11.22 | 11.21 |
| 1 | 0.3 | 39.32 | 37.32 | 30.32 | 22.31 | 14.3 | 9.28 | 7.25 | 6.22 | 7.2 |
| 1 | 0.1 | 6.33 | 6.33 | 6.32 | 5.33 | 4.32 | 4.28 | 3.26 | 3.22 | 3.2 |
| AMPLIFICATION FACTOR (Q) | 20 | | | | | | | | | |
| RESIDUAL MASS RATIO (M2/M1) | | 0.001 | 0.003 | 0.01 | 0.03 | 0.1 | 0.3 | 1 | 3 | 10 |
| RATIO OF MODAL TO RESIDUAL MASS | | | | | | | | | | |
| Source /M | Load m2/M2) | | | | | | | | | |
| 0.7 | 1 | 372.27 | 312.27 | 200.27 | 103.27 | 57.26 | 34.25 | 27.23 | 24.22 | 23.22 |
| 0.7 | 0.7 | 191.27 | 170.27 | 120.27 | 67.27 | 39.26 | 23.25 | 18.23 | 16.22 | 15.21 |
| [illegible] | [illegible] | [illegible] | 94 [illegible] | [illegible] | [illegible] | [illegible] | | | | |
| [illegible] | [illegible] | 38 [illegible] | 38 [illegible] | 33 [illegible] | [illegible] | [illegible] | [illegible] | | | |
| 0.7 | 0.1 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] |
| AMPLIFICATION FACTOR | | | | | | | | | | |
| RESIDUAL MASS RATIO (M2/M1) | | 0.001 | 0.003 | 0.01 | 0.03 | 0.1 | 0.3 | 1 | 3 | 10 |
| RATIO OF MODAL TO RESIDUAL MASS | | | | | | | | | | |
| Source (m1/M1) | Load (m2/M2) | | | | | | | | | |
| 0.5 | 1 | 377.24 | 319.24 | 211.24 | 117.24 | 59.23 | 39.23 | 27.22 | 24.22 | 22.22 |
| 0.5 | 0.7 | 193.24 | 172.24 | 125.24 | 74.24 | 39.23 | 25.23 | 19.22 | 15.22 | 15.21 |
| 0.5 | 0.5 | 102.24 | 95.24 | 74.24 | 48.24 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] |
| 0.5 | 0.3 | 39.24 | 38.24 | 33.24 | 24.24 | 15.23 | 10.23 | 8.21 | 8.2 | 7.2 |
| 0.5 | 0.1 | 6.25 | 6.24 | 6.24 | 6.24 | 5.24 | 4.23 | 3.22 | 4.19 | 4.18 |
| AMPLIFICATION FACTOR (Q) | 20 | | | | | | | | | |
| RESIDUAL MASS RATIO (M2/M1) | | 0.001 | 0.003 | 0.01 | 0.03 | 0.1 | 0.3 | 1 | 3 | 10 |
| RATIO OF MODAL TO RESIDUAL MASS | | | | | | | | | | |
| Source (m1/M1) | Load (m2/M2) | | | | | | | | | |
| 0.3 | 1 | 367.21 | 319.21 | 230.21 | 142.21 | 76.21 | 41.21 | 27.21 | [illegible] | [illegible] |
| 0.3 | 0.7 | 188.21 | 170.21 | 132.21 | 86.21 | 48.21 | 29.21 | 20.21 | 17.21 | 16.21 |
| 0.3 | 0.5 | 100.21 | 93.21 | 77.21 | 54.21 | 31.21 | 19.21 | 13.21 | 11.21 | 11.2 |
| 0.3 | 0.3 | 39.21 | 37.21 | 33.21 | 26.21 | 17.21 | 12.2 | 9.2 | 7.2 | 7.19 |
| 0.3 | 0.1 | 6.21 | 6.21 | 6.21 | 6.21 | 5.21 | 5.2 | 4.19 | 4.18 | 4.18 |
| AMPLIFICATION FACTOR | | | | | | | | | | |
| RESIDUAL MASS RATIO (M2/M1) | | 0.001 | 0.003 | 0.01 | 0.03 | 0.1 | 0.3 | 1 | 3 | 10 |
| RATIO OF MODAL RESIDUAL MASS | | | | | | | | | | |
| Source /M | oad 2/M | | | | | | | | | |
| 0.1 | 1 | 343.18 | 320.18 | 271.18 | 209.18 | 94.18 | 46.19 | 28.21 | 24.21 | 22.22 |
| 0.1 | 0.7 | 174.18 | 166.18 | 146.18 | 117.18 | 70.18 | 37.19 | 21.2 | 16.21 | 15.21 |
| 0.1 | 0.5 | 93.18 | 89.18 | 81.18 | 68.18 | 50.18 | 23.18 | 15.19 | 12.2 | 11.2 |
| [illegible] | [illegible] | 36 [illegible] | 36 [illegible] | [illegible] | 30 [illegible] | [illegible] | | | | |
| 0.1 | 0.1 | 6.18 | 6.18 | 6.18 | 6.18 | 5.18 | 5.18 | 4.18 | 4.18 | 3.19 |

**Table 1.** Q=20, Maximum Tuned Force Spectrum Normalized by Load Residual Mass Squared and Acceleration Test Specification (decimals indicate in sixteenths the ratio of load to source resonance frequency squared for tuned maximum force)

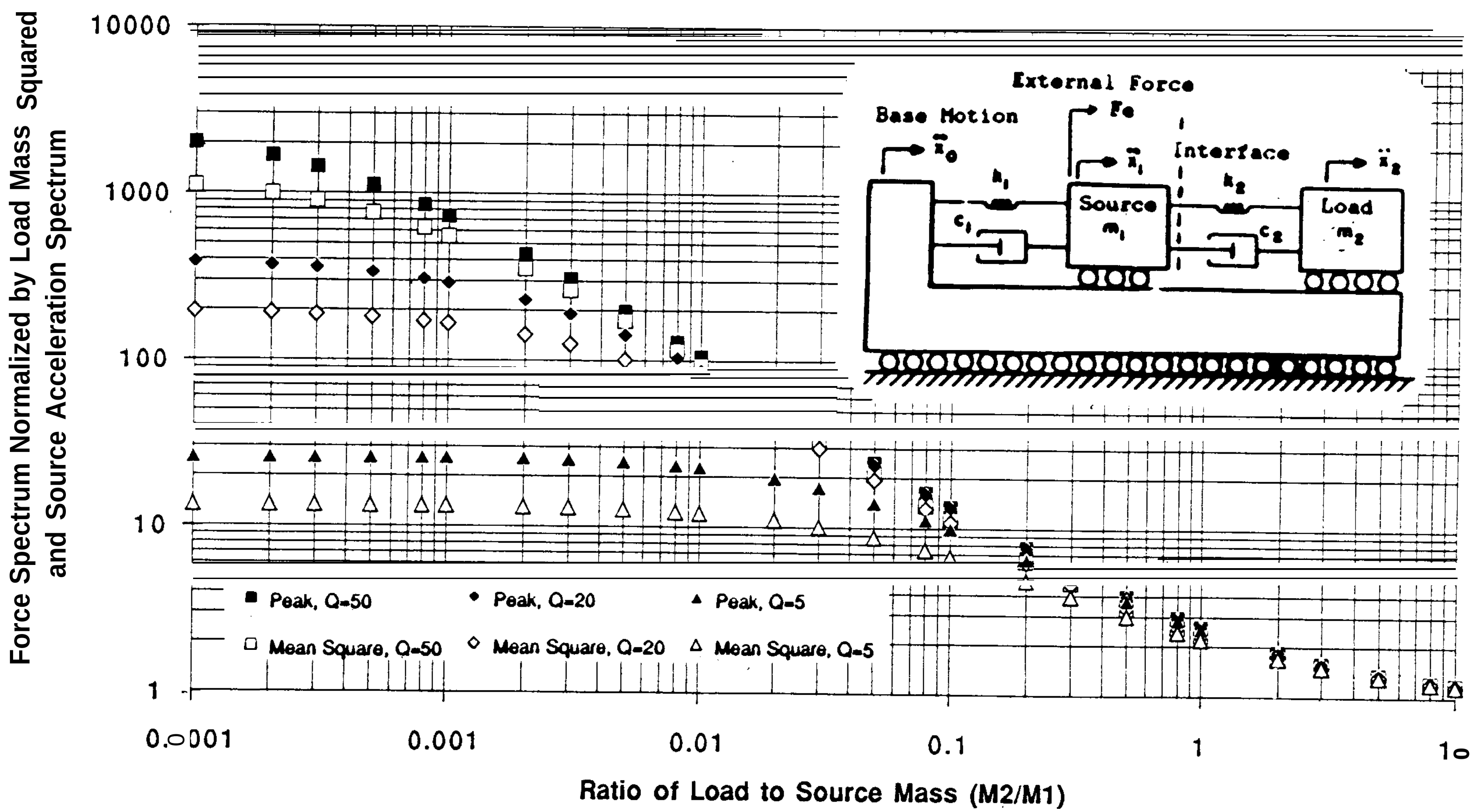
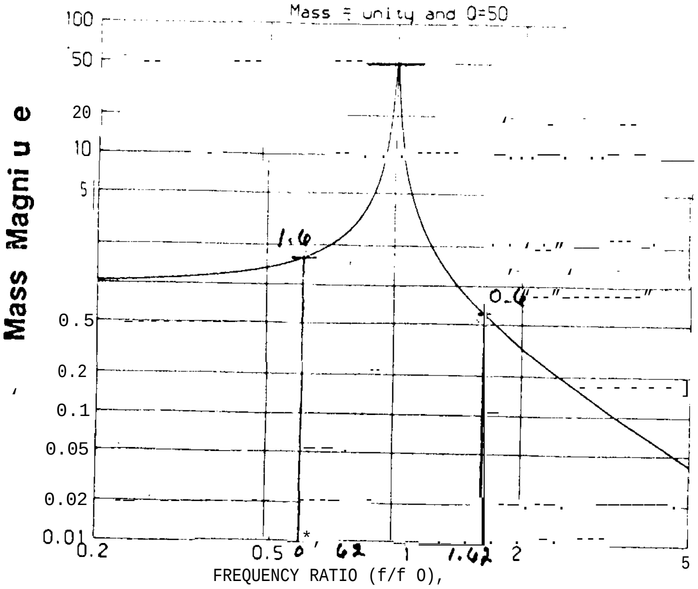
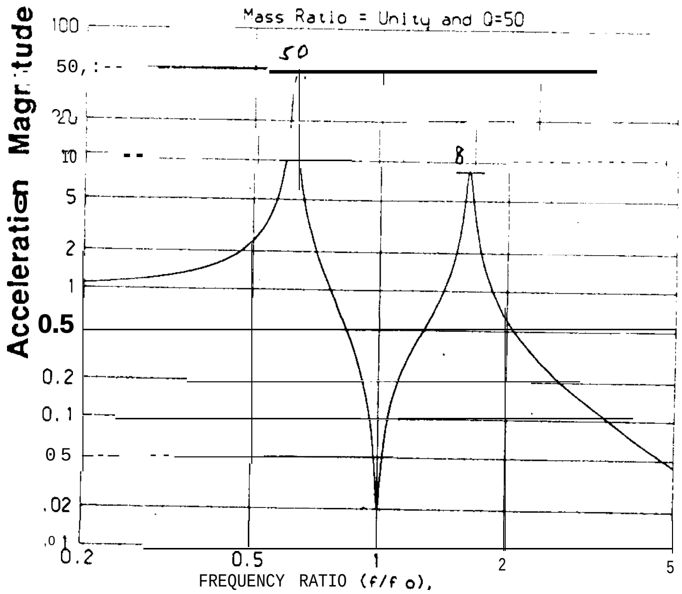


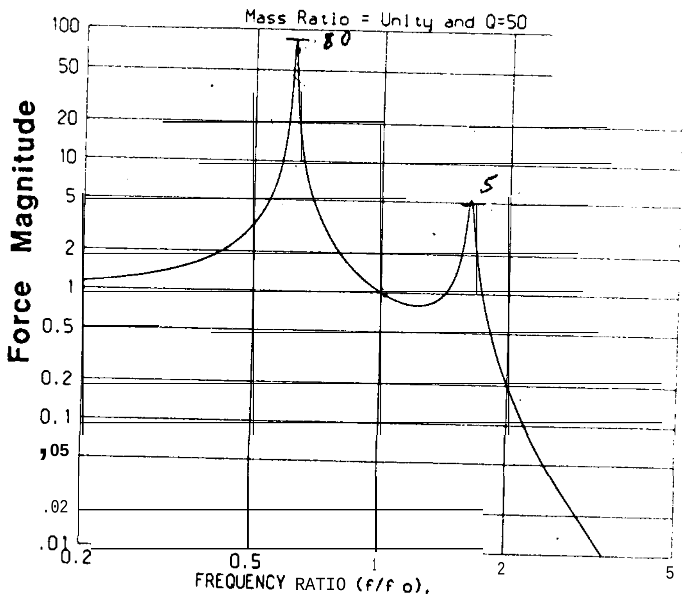

**Figure 1.** Comparison of Force Specra for TDFS Oscillators Calculated Using Mean-Square Response Ratios and Using Frequency Shift Method Peak Ratios




2a. LOAD DYNAMIC MASS, $M_2$




2b. INTERFACE ACCELERATION, A




2c. INTERFACE FORCE, F

**Figure 2. Load Dynamic Mass, Acceleration, and Force Frequency Response Functions for Identical Oscillator TDFS**

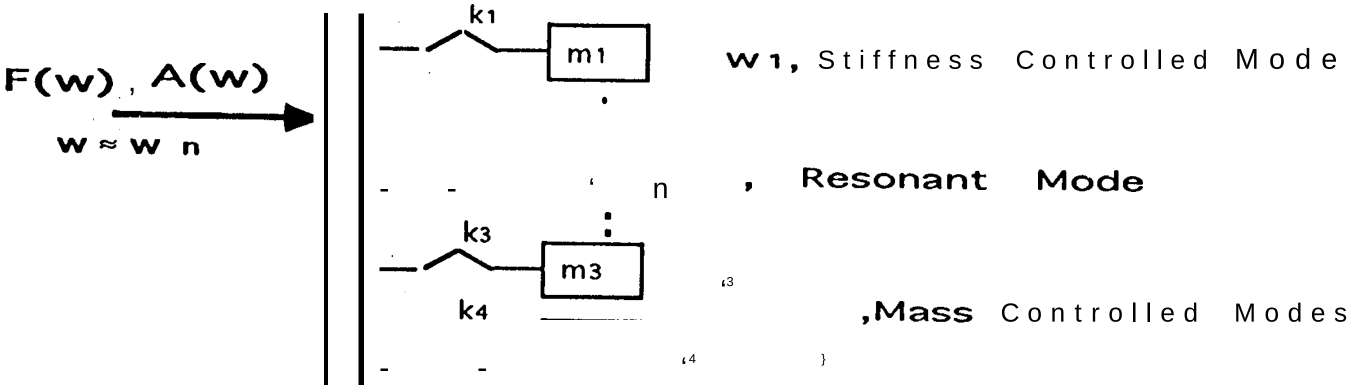



**Fig. 3a ASPARAGUS PATCH MODEL OF SOURCE OR LOAD**

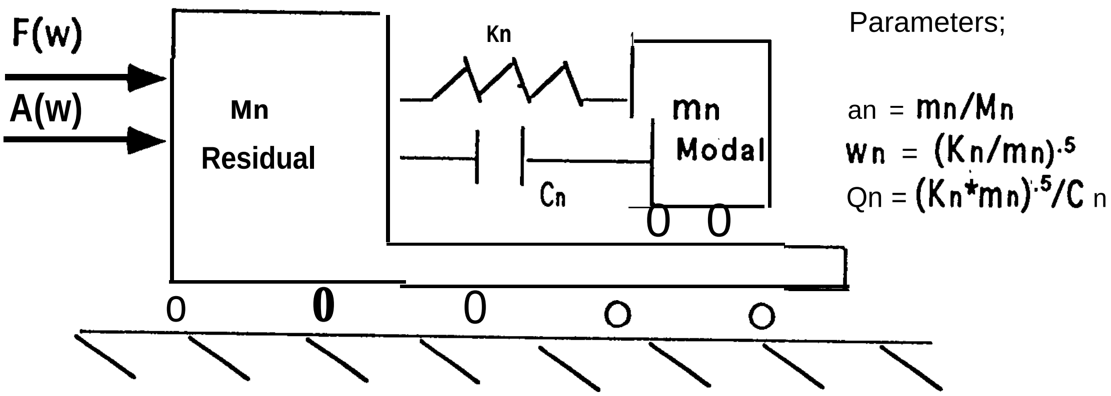



**Fig. 3b RESIDUAL AND MODAL MASS MODEL OF SOURCE OR LOAD**

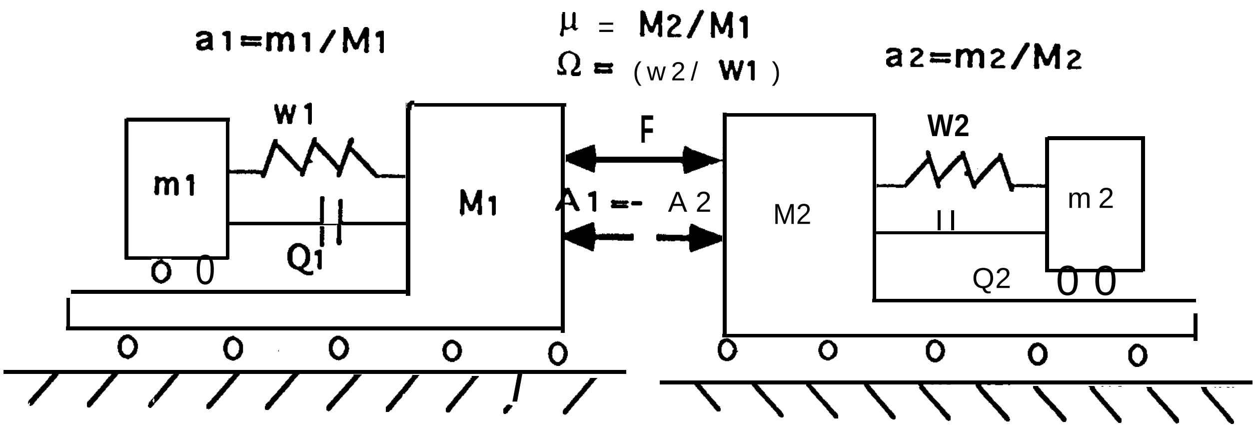



**Fig. 3c COUPLED TDFS RESIDUAL AND MODAL MASS MODEL**